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Hamiltonian Noether theorem for gauge systems and two time physics

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Abstract

The Noether theorem for Hamiltonian constrained systems is revisited. In particular, our review presents a novel method to show that the gauge transformations are generated by the conserved quantities associated with the first class constraints. We apply our results to the relativistic point particle, to the Friedberg *et al* model and, with special emphasis, to two time physics.

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1. Introduction

The main motivation of these notes is to revisit the Hamiltonian approach of the Noether theorem [1] in the case of singular systems. Our formalism is focused entirely on the Hamiltonian sector and we do not attempt to describe the corresponding Lagrangian sector in the sense of Gràcia and Pons [2–4] construction.

We work in a fundamental constrained Hamiltonian formalism [5–8], which is characterized by a first-order functional action defined on the phase-space variables q^i and p_j, with $i, j = 1, \ldots, n$. Consider the first class canonical Hamiltonian $H(q, p; t)$ and all the first class constraints of the system $\phi_\alpha(q, p; t)$ with their respective Lagrange multipliers $\lambda^\alpha(t)$. The first-order action is

$$S[q, p; \lambda^\alpha] = \int_{t_i}^{t_f} \mathrm{d}t[\dot{q}^i p_i - H(q, p; t) - \lambda^\alpha(t)\phi_\alpha(q, p; t)], \tag{1}$$

where $\dot{q}^i = \frac{\mathrm{d}}{\mathrm{d}t}q^i$ and $\alpha = 1, 2, \ldots, r$. Here, the Lagrange multipliers $\lambda^\alpha(t)$ are not regarded as dynamical degrees of freedom, but only as auxiliary variables which parametrize the gauge degrees of freedom of the system. In fact, it is not difficult to prove that if one considers the Lagrange multipliers as dynamical variables, then their associated canonical momenta π_α are first class constraints which only lead to arbitrary shifts in the Lagrange multipliers, in total agreement with their auxiliary character and do not act on the phase-space variables [6].

We shall assume that in this fundamental Hamiltonian formalism, all second class constraints, if any, have been solved and implemented in the dynamics of the system in such a way that only first class constraints are involved in the gauge invariance as well as on the dynamical evolution of the system [6, 7].

Our main task is to obtain Noether's first and second theorems [6] for gauge systems in the Hamiltonian sector (see [9–13]). In particular, when the second theorem is applied, we conclude that the conserved quantities are precisely the first class constraints. It is important to mention that, in the case of regular systems (free of constraints), all the well-known results are obtained.

We first apply our formalism to two examples: the relativistic point particle, and the Friedberg *et al* model [14] which has been studied and solved for the case of Gribov ambiguities in [15, 16]. We show that in these cases the conserved quantities are precisely the first class constraints.

It turns out that two time physics (see [17] and references therein) offers another interesting example for applying our formalism. The main reason is that in two time physics the variables q^i and p^j are unified in just one object x_a^i, with $a = 1, 2$, where $x_1^i \equiv q^i$ and $x_2^i \equiv p^i$, and consequently in the corresponding action the hidden symmetry $Sp(2, R)$ or $SL(2, R)$ becomes manifest (see [18–22]). Thus, we show that our formalism shed some new light on this hidden symmetry.

This work is organized as follows. In section 2, we discuss the Hamiltonian Noether theorem for gauge systems. In sections 3 and 4, we describe the Noether's first and second theorems respectively. In section 5, we apply our procedure to two examples: the relativistic scalar particle and the helix model of Friedberg *et al*. In section 6, we discuss two time physics from the point of view of our formalism. Finally, in section 7 we make some final remarks.

2. Hamiltonian Noether theorem for gauge systems

Let us first rewrite the action (1) in the form

$$S[q, p; \lambda^\alpha] = \int_{t_i}^{t_f} \mathrm{d}t[\dot{q}^i p_i - H_T], \tag{2}$$

where

$$H_T = H(q, p; t) + \lambda^\alpha(t)\phi_\alpha(q, p; t) \tag{3}$$

denotes the total Hamiltonian. Our aim is to see the consequences of applying to the action (2) the total variations:

$$\delta t = t'(t) - t, \tag{4}$$

$$\delta_\star q^i = q'^i(t') - q^i(t) = \delta q^i + \dot{q}^i \delta t, \tag{5}$$

$$\delta_\star p_i = p'_i(t') - p_i(t) = \delta p_i + \dot{p}_i \delta t \tag{6}$$

and

$$\delta_\star \lambda^\alpha = \lambda'^\alpha(t') - \lambda^\alpha(t) = \delta\lambda^\alpha + \dot{\lambda}^\alpha \delta t, \tag{7}$$

where $\delta q^i = q'^i(t) - q^i(t)$ and similar expressions hold for δp_i and $\delta\lambda^\alpha$. Observe that expression (5) for $\delta_\star q^i$ implies

$$\delta_\star \dot{q}^i = \delta\dot{q}^i + \ddot{q}^i \delta t. \tag{8}$$

It is important to remark that $\delta\dot{q}^i = \frac{\mathrm{d}}{\mathrm{d}t}\delta q^i$ but $\delta_\star\dot{q}^i \neq \frac{\mathrm{d}}{\mathrm{d}t}\delta_\star q^i$.

Invariance of the action (2) under total variations means that

$$\delta_\star S = \int_{t_i}^{t_f} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\delta_\star\Lambda(q, p), \tag{9}$$

where $\Lambda(q, p)$ is an arbitrary function. Thus, using transformations (4)–(7) we obtain

$$\delta_\star S = \int_{t_i}^{t_f} \mathrm{d}t\,\delta_\star[\dot{q}^i p_i - H_T] + \int_{t_i}^{t_f} \mathrm{d}t \frac{\mathrm{d}\delta t}{\mathrm{d}t}[\dot{q}^i p_i - H_T] = \int_{t_i}^{t_f} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\delta_\star\Lambda(q, p). \tag{10}$$

It is not difficult to show that expression (10) leads to

$$\int_{t_i}^{t_f} \mathrm{d}t \left\{ \frac{\mathrm{d}}{\mathrm{d}t} Q + \dot{q}^i \delta_\star p_i - \dot{p}_i \delta_\star q^i + \delta t \dot{H}_T - \delta_\star H_T \right\} = 0. \tag{11}$$

Here, the variable $Q = Q(q, p; t)$ is defined as

$$Q = \delta_\star q^i p_i - \delta t H_T - \delta_\star \Lambda. \tag{12}$$

By virtue of the definitions of the total variations (4)–(7), we find that relation (11) can also be written as

$$\int_{t_i}^{t_f} \mathrm{d}t \left\{ \frac{\mathrm{d}}{\mathrm{d}t} Q + \dot{q}^i \delta p_i - \dot{p}_i \delta q^i - \delta H_T \right\} = 0. \tag{13}$$

Let us write (13) in the form

$$A + B = 0, \tag{14}$$

where

$$A = \int_{t_i}^{t_f} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} Q \tag{15}$$

and

$$B = \int_{t_i}^{t_f} \mathrm{d}t \{\dot{q}^i \delta p_i - \dot{p}_i \delta q^i - \delta H_T\}. \tag{16}$$

Expression (14), or (13), offers three different possibilities, namely

(i) If $A = 0$ then (13) implies that $B = 0$.
(ii) If $B = 0$ then (13) implies that $A = 0$.
(iii) If neither A nor B are zero then (13) establishes that $A + B = 0$.

The first two cases are well known, but the third one seems to have passed unnoticed. In order to clarify these observations let us briefly discuss each one of these three cases. In the first case, we assume that the quantity Q satisfies the expression

$$Q|_{t_i}^{t_f} = 0, \tag{17}$$

which is equivalent to saying that $\int_{t_i}^{t_f} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} Q = 0$ or $A = 0$. Thus, for arbitrary variations, δq^i, δp_i and $\delta\lambda^\alpha$, (13) yields

$$\int_{t_i}^{t_f} \mathrm{d}t \left\{ \left(\dot{q}^i - \frac{\partial}{\partial p_i} H_T \right) \delta p_i + \left(-\dot{p}_i - \frac{\partial}{\partial q^i} H_T \right) \delta q^i - \delta\lambda^\alpha \phi_\alpha \right\} = 0, \tag{18}$$

and therefore we get the equations of motion:

$$\dot{q}^i = \frac{\partial}{\partial p_i} H_T = \{q^i, H_T\}, \tag{19}$$

$$\dot{p}_i = -\frac{\partial}{\partial q^i} H_T = \{p_i, H_T\} \tag{20}$$

and

$$\phi_\alpha = 0. \tag{21}$$

Here, the symbol $\{f, g\}$, for any functions f and g of the canonical variables q^i and p_i, stands for the usual Poisson bracket, that is

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}. \tag{22}$$

In the second case, we assume that the dynamical system satisfies equations of motion (19)–(21). This means that (18) follows, which means that $B = 0$. Therefore, from (13) we see that

$$\int \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} Q = 0. \tag{23}$$

Since by hypothesis the interval t_f–t_i is arbitrary, from (23) we have $\frac{\mathrm{d}}{\mathrm{d}t} Q = 0$ and therefore we find that Q is a conserved quantity.

The last possibility arises if we assume that neither (18) nor (23) hold, that is, we assume that A and B are different from zero. We shall show that in this case expression (13) implies that Q is the generator of canonical transformations. For this purpose let us first compute $\frac{\mathrm{d}}{\mathrm{d}t} Q$. Since $Q = Q(q, p; t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} Q = \frac{\partial Q}{\partial q^i}\dot{q}^i + \frac{\partial Q}{\partial p_i}\dot{p}_i + \frac{\partial Q}{\partial t}. \tag{24}$$

Thus, for an undefined interval t_f–t_i, (13) becomes

$$\int \mathrm{d}t \left\{ \frac{\partial Q}{\partial q^i}\dot{q}^i + \frac{\partial Q}{\partial p_i}\dot{p}_i + \frac{\partial Q}{\partial t} + \dot{q}^i \delta p_i - \dot{p}_i \delta q^i - \delta H_T \right\} = 0, \tag{25}$$

which can be rewritten as

$$\int \mathrm{d}t \left\{ \left(\frac{\partial Q}{\partial q^i} + \delta p_i \right) \dot{q}^i + \left(\frac{\partial Q}{\partial p_i} - \delta q^i \right) \dot{p}_i + \frac{\partial Q}{\partial t} - \delta H_T \right\} = 0, \tag{26}$$

or

$$\int \left\{ \left(\frac{\partial Q}{\partial q^i} + \delta p_i \right) \mathrm{d}q^i + \left(\frac{\partial Q}{\partial p_i} - \delta q^i \right) \mathrm{d}p_i + \left(\frac{\partial Q}{\partial t} - \delta H_T \right) \mathrm{d}t \right\} = 0. \tag{27}$$

If we now define the quantity

$$\omega = \left(\frac{\partial Q}{\partial q^i} + \delta p_i \right) \mathrm{d}q^i + \left(\frac{\partial Q}{\partial p_i} - \delta q^i \right) \mathrm{d}p_i + \left(\frac{\partial Q}{\partial t} - \delta H_T \right) \mathrm{d}t, \tag{28}$$

we observe that (27) gives

$$\int \omega = 0. \tag{29}$$

From (28) we observe that ω may admit an interpretation of 1-form. Thus, under usual assumptions (29) implies that ω is an exact form which means that

$$\omega = \mathrm{d}f, \tag{30}$$

where f is an arbitrary 0-form.

We shall assume that $f = f(q, p)$. From (28) and (27) we see that

$$\frac{\partial Q}{\partial t} - \delta H_T = 0. \tag{31}$$

Thus, considering (31) we discover that expressions (27) and (30) yield

$$\left(\frac{\partial Q'}{\partial q^i} + \delta p_i\right) \mathrm{d}q^i + \left(\frac{\partial Q'}{\partial p_i} - \delta q^i\right) \mathrm{d}p_i = 0, \tag{32}$$

where

$$Q' = Q + f. \tag{33}$$

Since $\mathrm{d}q^i$ and $\mathrm{d}p_i$ are 1-form bases, we find that (32) implies

$$\delta q^i = \frac{\partial Q'}{\partial p_i} = \{q^i, Q'\} \tag{34}$$

and

$$\delta p_i = -\frac{\partial Q'}{\partial q^i} = \{p_i, Q'\}. \tag{35}$$

Thus, we have shown that up to an arbitrary function f the quantity Q, which is a conserved quantity when the equations of motion are satisfied, is the generator of canonical transformations.

In order to clarify the meaning of expression (31), we investigate the consequences of invariances under gauge transformations, i.e., we consider the particular case

$$Q' = \xi^\alpha(t)\phi_\alpha(q, p; t), \tag{36}$$

where the quantities $\xi^\alpha(t)$ are infinitesimal parameters associated with the first class constraints $\phi_\alpha(q, p; t)$. Moreover, since we are dealing (by assumption) with only first class constraints $\phi_\alpha(q, p; t)$, we can write (see [6, 23])

$$\{H, \phi_\alpha\} = V_\alpha^\beta \phi_\beta \tag{37}$$

and

$$\{\phi_\alpha, \phi_\beta\} = C_{\alpha\beta}^\gamma \phi_\gamma, \tag{38}$$

where V_α^β and $C_{\alpha\beta}^\gamma$ are structure 'constants'. Then, (31), (34)–(36) lead to

$$\delta\lambda^\alpha \phi_\alpha = \left(\dot{\xi}^\alpha - \xi^\beta V_\beta^\alpha - \xi^\beta \lambda^\gamma C_{\beta\gamma}^\alpha\right)\phi_\alpha. \tag{39}$$

Considering the first class constraints $\phi_\alpha(q, p; t)$ are independent functions we get that expression (39) implies

$$\delta\lambda^\alpha = \dot{\xi}^\alpha - \xi^\beta V_\beta^\alpha - \xi^\beta \lambda^\gamma C_{\beta\gamma}^\alpha, \tag{40}$$

which is the usual result for the transformations of the Lagrange multipliers λ^α under gauge transformations generated by the first class constraints $\phi_\alpha(q, p; t)$ (see [6, 7, 23]).

3. Noether's first theorem

We now consider the consequences of previous discussion for the particular case of transformations that define a simply connected continuous group. In other words, we are interested in studying the so-called first Noether theorem which refers to the invariance of the action (2) under global transformations. Of course, these transformations are not associated with a gauge physical system because this arises when one assumes local transformations. Thus, we consider the transformations

$$\begin{aligned} \delta t &= \xi^\alpha \chi_\alpha(t), & \delta q^i &= \xi^\alpha \varphi_\alpha^i(q, p; t), \\ \delta p_i &= \xi^\alpha \psi_{i\alpha}(q, p; t), & \delta\Lambda &= \xi^\alpha \Lambda_\alpha(q, p; t), \end{aligned} \tag{41}$$

with the abbreviations

$$\varphi^i_\alpha = \{q^i, \phi_\alpha\}, \qquad \psi_{i\alpha} = \{p_i, \phi_\alpha\}, \qquad \Lambda_\alpha = \{\Lambda, \phi_\alpha\}. \tag{42}$$

Here, ξ^α, with $\alpha = 1, 2, \ldots, r$, is an infinitesimal constant parameter spanning infinitesimal group transformations, with r as the dimension of such a group.

Using (41) we find that expression (13) gives

$$\int_{t_i}^{t_f} \mathrm{d}t\, \xi^\alpha \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left[\varphi^i_\alpha p_i + \dot{q}^i p_i \chi_\alpha - \chi_\alpha H_T - \Lambda_\alpha - \dot{\Lambda}\chi_\alpha\right] + \left(\dot{q}^i - \frac{\partial}{\partial p_i} H_T\right)\psi_{i\alpha} + \left(-\dot{p}_i - \frac{\partial}{\partial q^i} H_T\right)\varphi^i_\alpha - \left(V^\beta_\alpha - \lambda^\gamma C^\beta_{\alpha\gamma}\right)\phi_\beta \right\} = 0. \tag{43}$$

Thus, according to our discussion of the previous section we find that relation (43) determines that, up to an arbitrary function, the quantity $Q = \xi^\alpha Q_\alpha$, where

$$Q_\alpha = \varphi^i_\alpha p_i + \dot{q}^i p_i \chi_\alpha - \chi_\alpha H_T - \Lambda_\alpha - \dot{\Lambda}\chi_\alpha \tag{44}$$

are the r-conserved Noether charges that in turn generate the transformations (41).

4. Noether's second theorem

As a second application, we now consider the case in which the parameters of transformation ξ^α are functions of the time t. In addition, we assume that the corresponding gauge transformations are generated by the first class constraints ϕ_α [6] (see [9, 10] for details),

$$\begin{aligned} \delta t &= \xi^\alpha(t)\chi_\alpha(t), & \delta q^i &= \xi^\alpha(t)\varphi^i_\alpha(q, p; t), \\ \delta p_i &= \xi^\alpha(t)\psi_{i\alpha}(q, p; t), & \delta\Lambda &= \xi^\alpha(t)\Lambda_\alpha(q, p; t), \\ \delta\lambda^\alpha &= \dot{\xi}^\alpha(t) - \xi^\beta(t)V^\alpha_\beta - \xi^\beta(t)\lambda^\gamma C^\alpha_{\beta\gamma}, \end{aligned} \tag{45}$$

where we used the definitions (42). It is important to mention that, in these expressions, we have not considered other possible derivatives of $\xi^\alpha(t)$. Nevertheless, the generalization to such cases seems to be straightforward.

The substitution of relations (45) into expression (13) yields

$$\int_{t_i}^{t_f} \mathrm{d}t \left\{ \frac{\mathrm{d}}{\mathrm{d}t}[\xi^\alpha(t)Q_\alpha(q, p; t)] + \xi^\alpha(t)\left[\left(\dot{q}^i - \frac{\partial}{\partial p_i}(H + \lambda^\rho\phi_\rho)\right)\psi_{i\alpha} + \left(-\dot{p}_i - \frac{\partial}{\partial q^i}(H + \lambda^\rho\phi_\rho)\right)\varphi^i_a\right] - \dot{\xi}^\alpha(t)\phi_\alpha - \xi^\alpha V^\rho_\alpha\phi_\rho - \xi^\alpha\lambda^\gamma C^\rho_{\alpha\gamma}\phi_\rho \right\} = 0, \tag{46}$$

with

$$Q_\alpha = \varphi^i_a p_i + \dot{q}^i p_i \chi_\alpha - \chi_\alpha(H + \lambda^\rho\phi_\rho) - \Lambda_\alpha. \tag{47}$$

Thus, according to the discussion in section 2, the Q_α given in (47) can be associated with the conserved charges or the generator of the gauge transformations (45) depending on whether the corresponding equations of motion of the gauge physical system are satisfied or not respectively.

5. Relativistic point particle and Friedberg *et al* model

Example 1: The relativistic scalar point particle. The general covariant Hamiltonian formulation of the free relativistic point particle of mass m_0 is provided by the phase space (x^μ, p_ν), with Poisson brackets

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \tag{48}$$

and the first class constraint

$$\phi = \tfrac{1}{2}\left(p^2 + m_0^2\right). \tag{49}$$

The total Hamiltonian H_T is given by

$$H_T = \frac{\lambda}{2}\left(p^2 + m_0^2\right), \tag{50}$$

where λ is the Lagrange multiplier associated with the first class constraint (49).

The corresponding fundamental first-order action is

$$S[x, p; \lambda] = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left[\dot{x}^\mu p_\mu - \frac{\lambda}{2}\left(p^2 + m_0^2\right)\right]. \tag{51}$$

This action is invariant (up to a surface term) under the transformation generated by the first class constraint

$$\delta x_\mu = \{x_\mu, \xi\phi\} = \xi p_\mu, \qquad \delta p^\mu = \{p^\mu, \xi\phi\} = 0, \qquad \delta\lambda = \dot{\xi}. \tag{52}$$

In fact, using (52) we get

$$\delta S = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\xi}{2}\left(p^2 - m_0^2\right)\right], \tag{53}$$

which leads to the surface term

$$\delta\Lambda = \xi\Lambda_1 = \frac{\xi}{2}\left(p^2 - m_0^2\right). \tag{54}$$

Thus, according to (41) we see that from (52) we can conclude that $\chi_1 = 0$, $\varphi_1^\mu = p^\mu$, $\psi_{\mu 1} = 0$. Using these results and Λ_1 given in (54) we find that expression (44) implies that our conserved quantity is

$$Q = \xi\left(\varphi_1^\mu p_\mu - \Lambda_1\right) = \frac{\xi}{2}\left(p^2 + m_0^2\right), \tag{55}$$

as expected.

Example 2: The Friedberg *et al* model. The helix model of Friedberg *et al* [14] (see also [15, 16]) can be described in terms of the fundamental Hamiltonian first-order action,

$$S = \int_{t_i}^{t_f} \mathrm{d}t\left[\dot{x}p_x + \dot{y}p_y + \dot{z}p_z - \frac{1}{2}\left[p_x^2 + p_y^2 + p_z^2\right] - U(x, y) - \lambda(p_z + g(xp_y - yp_x))\right], \tag{56}$$

where (x, y, z) and (p_x, p_y, p_z) stand for three-dimensional coordinates and canonical momenta respectively. Where $U(x, y) = U(x^2+y^2)$, and λ is a Lagrange multiplier associated with the first class constraint $\phi = p_z + g(xp_y - yp_x)$, where g denotes a coupling constant.

This action is invariant under the infinitesimal gauge transformations

$$\delta x = -\alpha y, \qquad \delta y = +\alpha x, \qquad \delta z = +\frac{1}{g}\alpha \tag{57}$$

and

$$\delta p_x = -\alpha p_y, \qquad \delta p_y = +\alpha p_x, \qquad \delta p_z = 0. \tag{58}$$

Furthermore, we have

$$\delta\lambda = +\frac{1}{g}\dot{\alpha}. \tag{59}$$

Thus, from (57)–(59) we find the following identifications: $\xi^1(t) = \frac{1}{g}\alpha(t), \chi_1 = 0, \varphi_1^1 = -gy, \varphi_1^2 = +gx$ and $\varphi_1^3 = 1$ as well as $\psi_{11} = -gp_y, \psi_{21} = +gp_x$ and $\psi_{31} = 0$. Observe that, in this case, the variation of the action is exactly zero and there is no need for the surface term as in the previous example. With the above ingredients, by direct substitution in (44), we get

$$\begin{aligned} Q_1 &= \varphi_1^i p_i + 0, \\ &= -gyp_x + gxp_y + 1p_z, \\ &= p_z + g(xp_y - yp_x), \end{aligned} \tag{60}$$

which is precisely the first class constraint of the physical system whose motion is governed by the action (56).

6. Two time physics

Two time physics is described by the action [17] (see also [18–22])

$$S = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left(\frac{1}{2}\varepsilon^{ab}\dot{x}_a^\mu x_b^\nu \eta_{\mu\nu} - H\left(x_a^\mu\right) \right), \tag{61}$$

where $\eta_{\mu\nu}$ is a flat metric whose signature will be determined below. Up to a total derivative this action is equivalent to the first-order action

$$S = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau (\dot{x}^\mu p_\mu - H(x, p)), \tag{62}$$

where

$$x^\mu = x_1^\mu, \qquad p^\mu = x_2^\mu. \tag{63}$$

For a relativistic point particle, one chooses H as $H_T = \lambda\left(p^\mu p_\mu + m_0^2\right)$ (see example 1 in section 5) or

$$H_T = \lambda(p^\mu p_\mu), \tag{64}$$

in the massless case. Observing that the first term in the action (61) has the manifest $Sp(2, R)$ (or $SL(2, R)$) invariance, we find that these choices for H spoil such a symmetry for the entire action (61).

It turns out that the simplest possible choice for H which maintains the symmetry $Sp(2, R)$ is

$$H = \tfrac{1}{2}\lambda^{ab} x_a^\mu x_b^\nu \eta_{\mu\nu}, \tag{65}$$

where $\lambda^{ab} = \lambda^{ba}$ is a Lagrange multiplier. One may also think in the 'massive' case that

$$H = \tfrac{1}{2}\lambda^{ab}\left(x_a^\mu x_b^\nu \eta_{\mu\nu} + m_{ab}^2\right), \tag{66}$$

with $m_{11}^2 = -R^2, m_{22}^2 = m_0^2$ and $m_{12}^2 = 0$, but one can see that the mass term m_{ab}^2 breaks the $Sp(2, R)$ symmetry. Considering (65) the action (61) becomes

$$S = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left(\frac{1}{2}\varepsilon^{ab}\dot{x}_a^\mu x_b^\nu \eta_{\mu\nu} - \frac{1}{2}\lambda^{ab} x_a^\mu x_b^\nu \eta_{\mu\nu} \right). \tag{67}$$

Arbitrary variations of λ^{ab} in (67) lead to the constraint

$$\Omega_{ab} = x_a^\mu x_b^\nu \eta_{\mu\nu} = 0, \tag{68}$$

which turns out to be first class.

In terms of the notation (63) we find that expression (68) gives (see [24])

$$x^\mu x_\mu = 0, \tag{69}$$

$$x^\mu p_\mu = 0 \tag{70}$$

and

$$p^\mu p_\mu = 0. \tag{71}$$

While in the 'massive' case (66) leads to

$$x^\mu x_\mu - R^2 = 0, \tag{72}$$

$$x^\mu p_\mu = 0 \tag{73}$$

and

$$p^\mu p_\mu + m_0^2 = 0. \tag{74}$$

The key point in two time physics comes from the observation that if $\eta_{\mu\nu}$ corresponds to just one time, that is, if $\eta_{\mu\nu}$ has the signature $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$ then from (69)–(71) it follows that p^μ is parallel to x^μ and therefore the angular momentum

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \tag{75}$$

associated with the Lorentz symmetry of (67) should vanish, which is, of course, an unlikely result. Thus, if we impose the condition $L^{\mu\nu} \neq 0$ and the constraints (69)–(71), we find that the signature of $\eta_{\mu\nu}$ should be at least of the form $\eta_{\mu\nu} = \mathrm{diag}(-1, -1, 1, \ldots, 1)$. In other words, only with two times are the constraints (69)–(71) consistent with the requirement $L^{\mu\nu} \neq 0$ (see [25–27]). In principle, we can assume that the number of times is greater than 2, but then one does not have enough constraints to eliminate all the possible ghosts.

With these observations at hand, we shall now proceed to generalize the action (2) in the form

$$S[x_a^\mu; \lambda] = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left[\frac{1}{2} \varepsilon^{ab} \dot{x}_a^\mu x_b^\nu \eta_{\mu\nu} - H_T \right]. \tag{76}$$

Here

$$H_T = H\left(x_a^\mu; \tau\right) - \lambda^{bc}(\tau)\phi_{bc}\left(x_a^\mu; \tau\right). \tag{77}$$

We are assuming that $\phi_{bc} = \phi_{cb}$ denotes a generalization of the first class constraint Ω_{ab} (see expression (68)).

Consider the transformations

$$\delta\tau = \tau'(\tau) - \tau, \qquad \delta_\star x_a^\mu = x_a'^\mu(\tau') - x_a^\mu(\tau) = \delta x_a^\mu + \dot{x}_a^\mu \delta\tau, \tag{78}$$
$$\delta_\star \lambda_{ab} = \lambda'_{ab}(\tau') - \lambda_{ab}(\tau) = \delta\lambda_{ab} + \dot{\lambda}_{ab}\delta\tau,$$

where $\delta x_a^\mu = x_a'^\mu(\tau) - x_a^\mu(\tau)$ and a similar expression for $\delta\lambda_{ab}$ holds. The expression for $\delta_\star x_a^\mu$ implies

$$\delta_\star \dot{x}_a^\mu = \delta\dot{x}_a^\mu + \ddot{x}_a^\mu \delta\tau. \tag{79}$$

We find that invariance of the action (76) under the transformations (78) gives

$$\delta_\star S = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \delta_\star \left[\frac{1}{2}\varepsilon^{ab}\dot{x}_a^\mu x_b^\nu \eta_{\mu\nu} - H_T\right] + \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \frac{\mathrm{d}\delta\tau}{\mathrm{d}\tau}\left[\frac{1}{2}\varepsilon^{ab}\dot{x}_a^\mu x_b^\nu \eta_{\mu\nu} - H_T\right]$$
$$= \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\delta_\star \Lambda\left(x_a^\mu;\tau\right), \tag{80}$$

where $\Lambda\left(x_a^\mu;\tau\right)$ is an arbitrary function. It is not difficult to show that this variation of the action S can be reduced to the form

$$\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{2}\varepsilon^{ab}\delta_\star x_a^\mu x_b^\nu \eta_{\mu\nu} - \delta\tau H_T - \delta_\star\Lambda\right] + \varepsilon^{ab}\dot{x}_a^\mu \delta x_b^\nu \eta_{\mu\nu} + \delta\tau \dot{H}_T - \delta_\star H_T\right\} = 0. \tag{81}$$

If we now define the variable $Q = Q\left(x_a^\mu;\tau\right)$ as

$$Q = \tfrac{1}{2}\varepsilon^{ab}\delta_\star x_a^\mu x_b^\nu \eta_{\mu\nu} - \delta\tau H_T - \delta_\star\Lambda, \tag{82}$$

then we find that (81) can be written as

$$\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left\{\frac{\mathrm{d}}{\mathrm{d}\tau}Q + \varepsilon^{ab}\dot{x}_a^\mu \delta_\star x_b^\nu \eta_{\mu\nu} + \delta\tau \dot{H}_T - \delta_\star H_T\right\} = 0 \tag{83}$$

or

$$\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left\{\frac{\mathrm{d}}{\mathrm{d}\tau}Q + \varepsilon^{ab}\dot{x}_a^\mu \delta x_b^\nu \eta_{\mu\nu} - \delta H_T\right\} = 0. \tag{84}$$

These expressions are of course the analogue of (11) or (13) respectively. Thus, following a similar procedure as in section 2, we can prove that Q plays a double role: a conserved quantity or a generator of canonical transformations depending on whether the Hamilton equations of motion hold or not.

Let us apply these results to two time physics. First we observe that in terms of coordinates x_a^μ the definition (22) for the Poisson brackets become

$$\{f,g\} = \varepsilon_{ab}\frac{\partial f}{\partial x_a^\mu}\frac{\partial g}{\partial x_{b\mu}}, \tag{85}$$

for any canonical functions $f\left(x_a^\mu\right)$ and $g\left(x_a^\mu\right)$. Thus, we find

$$\left\{x_a^\mu, x_b^\nu\right\} = \varepsilon_{ab}\eta^{\mu\nu}. \tag{86}$$

From this result it is straightforward to check that the constraint Ω_{ab} given in (68) gives

$$\{\Omega_{ab}, \Omega_{cd}\} = C_{abcd}^{ef}\Omega_{ef}, \tag{87}$$

where the structure constants C_{abcd}^{ef} are given by

$$C_{abcd}^{ef} = \tfrac{1}{2}\left[\varepsilon_{ac}\left(\delta_b^e\delta_d^f + \delta_d^e\delta_b^f\right) + \varepsilon_{ad}\left(\delta_b^e\delta_c^f + \delta_c^e\delta_b^f\right) + \varepsilon_{bc}\left(\delta_a^e\delta_d^f + \delta_d^e\delta_a^f\right) + \varepsilon_{bd}\left(\delta_a^e\delta_c^f + \delta_c^e\delta_a^f\right)\right]. \tag{88}$$

Expression (87) establishes that the constraint Ω_{ab} is in fact a first class constraint.

Consider the variable

$$Q = \xi^{ab}(\tau)\Omega_{ab}, \tag{89}$$

where $\xi^{ab} = \xi^{ba}$ are infinitesimal parameters. According to the previous discussion, this variable should be a conserved quantity or the generator of gauge transformations depending on whether the equations of motion are satisfied or not. In fact, using the formulae

$$\delta x_a^\mu = \left\{x_a^\mu, Q\right\} \tag{90}$$

and

$$\delta_\star H_T - \frac{\partial}{\partial t} Q = 0, \tag{91}$$

which can be derived from (84) when the equations of motion are not satisfied, we obtain that the constraint Ω_{ab} generates the transformations

$$\delta x_a^\mu = \varepsilon_{ab} \xi^{bc} x_c^\mu \tag{92}$$

and

$$\delta \lambda^{ab} = \dot{\xi}^{ab} - \xi^{ef} \lambda^{cd} C^{ab}_{efcd}. \tag{93}$$

We recognize in expression (92) the infinitesimal transformation associated with the group $Sp(2, R) \cong SL(2, R)$ with infinitesimal parameter $\varsigma_a^c = \varepsilon_{ab}\xi^{bc}$. Thus, we have proved that if the Lagrange multipliers variation $\delta\lambda^{ab}$ is given by (93) then the action (67) is invariant under the $Sp(2, R)$ gauge transformation (92). The remarkable fact is that this $Sp(2, R)$ invariance of the action (67) is generated by the conserved quantity (89) corresponding to the first class constraint Ω_{ab}.

7. Final remarks

In this work we revisited Noether's first and second theorems. One of the novel features of our presentation is that the canonical transformations can be obtained directly from the action when the time derivative of quantity Q is different from zero and the equations of motion are not satisfied. We proved that our method may reveal hidden symmetries in specific cases.

As an application of our formalism we considered the cases of a relativistic point particle and the Friedberg *et al* model. On the other hand, since in two time physics the phase space has a unified character in the sense that the spacetime and the momentum space are put together at the same level, we found that an application of our formalism in this context requires a generalization of the usual Noether's procedure. As a consequence of such a generalization, we show explicitly how the gauge transformations for the coordinates and momenta, generated by the Hamiltonian constraint associated with two time physics, also exhibit a unified character. Moreover, using our method we have proved that the conserved quantity Q given in (89) written in terms of the first class constraint Ω_{ab} generates the gauge symmetry of the action (67), clarifying further the origin of the manifest gauge $Sp(2, R)$ symmetry.

The simplest possible action (67) corresponds to the 'free' theory, in the sense of the flat metric $\eta_{\mu\nu}$. Further generalization to an interacting theory is possible by including gravitational background, gauge fields, other potentials and higher spin fields [19]. In those generalized cases the constraint $\Omega_{ab} = x_a^\mu x_b^\nu \eta_{\mu\nu}$ is extended by including a curved metric $g_{\mu\nu}(x)$, gauge potential $A_\mu(x)$, a potential $U(x)$ and higher spin fields. In order to ensure the invariance $Sp(2, R)$ of the corresponding action, the constraint (87) is imposed as a differential constraint. An interesting aspect is that those generalized cases of two time physics give all possible background fields in one time physics (see [19, 21] for details).

It may be interesting for further research to analyse the gauge fixing and quantization of constrained Hamiltonian systems [28] from the point of the present formalism. Another possible application of our formalism is related to the connection between oriented matroid theory [29] (for application of matroid theory on high energy physics, see [30–32]) and two time physics. In fact, via the chirotope concept in [25–27] it was shown that there is a deep connection between oriented matroid theory and two time physics. Therefore it seems attractive to establish a connection between the concepts of chirotope and gauge symmetry using the Noether theorem as discussed in this work. Recently, a complete analysis of Dirac's conjecture has been given [33]. In this analysis the Legendre map from the tangent bundle and the cotangent bundle (phase space) plays an essential role. In turn, the cotangent bundle provides an example of geometrical structure of fibre bundles. It may be interesting to find a connection between such a geometrical structures and the present formalism.

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Note added. After we finished this work, we received an e-mail from I Bars in which he called our attention to the footnote 3 in [34]. In fact, in this footnote there is also a discussion about the Q-conserved quantity. However, such a discussion is made at the level of the Lagrangian and not on the full action. Moreover, Q is assumed, from the beginning, to be a generator of the canonical transformations.

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